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Flowery math: a case for heterodiscoursia in mathematics problems solving in recognition of students' authorial agency

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In an innovative, progressive school, students were asked to solve a fairly routine mathematical problem using real money in a "real-world" scenario. Even though the school values students' ideas, the reaction of the teacher to one student's alternative modelling of the problem suggests that he was expecting a particular answer to be provided using routine mathematical models and thinking while not being interested in exploring the student's unexpected alternative. We place his reasoning for doing so within broad pedagogical discourses that we think define the "allowable" responses of teachers and students in ways that inhibit meaning-making for both. These broad discourses are defined as the progressive constructivist approach, the scaffolding discursive approach, the situation modelling approach and the dialogic approach. We consider the advantages and the potential consequences each might bring to the case. We suggest that extensive consideration of pedagogical discourses in mathematics classes must be reconsidered both for how we understand students' mathematical meaning-making and how we construct student agency in relationship to culture, whether as apprentices or authors.

Keywords: dialogic approach; mathematics; authorial agency; discourses; heterodiscoursia

In this paper, we present an opportunistic research case that deeply puzzled us: opportunistic because this case emerged within another research project, puzzling because neither we nor the teacher initially understood what defined the case but felt its importance. We first describe the case as we captured it on video, our puzzlement with it and our method for gathering data. Our means of analysis takes a sociocultural approach and interprets the case through four broad pedagogic discourses, or approaches, to mathematics instruction. We discuss the relative ways in which these discourses inhibit both the student and the teacher in their "allowable" responses to the problem and the search for solutions. As Bakhtin has said, "We speak only in definite speech genres, that is, all our utterances have definite and relatively stable typical *forms of construction of the whole*" (1986, p. 78). We suggest that an important area for research is how stable constructions of pedagogical speech genres constrain student meaning-making, and we suggest this is important in order to be better understand students' mathematical thinking.

A puzzling case of flowery mathematics

Our case is opportunistic in that it arose as a part of another study focusing on a teacher's pedagogical decision-making. We videotaped a mathematics lesson in a mixed third and

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fourth grade in a small, private, innovative, collaborative, private, K-8 school, Creative Learning Center (CLC). In our footage, we discovered a perplexing pedagogical case. In the following three short exchanges, we track the difference between the teacher and three students' interpretation of a mathematics problem; a difference that seemed unsatisfactorily unresolved and unexplained, for students and teacher. We then interviewed the teacher to understand how broad pedagogical discourses guided his decision-making, and finally consider how these discourses suggest teachers should approach this difference.

First, a play-by-play of three student–teacher exchanges. In the following paragraphs, we detail a brief narrative of the pertinent data. In our view, these data are a drop of water in an ocean of recognizable teacher action, and for this reason, unremarkable at first glance.

John,[1] an experienced, innovative and energetic teacher, gives his students a pile of mixed coins: cents, nickels, dimes and quarters. He invites the students to divide a dollar's worth among three imagined friends evenly, "Divide 100 cents among yourself and two imaginary friends then tell me the answer you get". He suggests selecting a dollar from the pile of the coins of different denominations first, before attempting to divide. The children in this classroom are free to talk with each other; walk around in the classroom; choose a place to work; observe each other's work or choose to work in small groups. The teacher begins browsing group-to-group and student-to-student, observing, helping, answering children's questions and guiding them in how to group the coins. Some children struggled to form three equal groups of coins; some did not know what to do with the remaining penny left from the division problem; and some did not know how to start solving the problem.

At one point, John comes to two boys working on solving the problem. He notices that Paul has formed six quarters around one quarter, creating a symmetrical eye-pleasing figure (in our perception). Paul smiles at his achievement and points at the composition with his finger to his friend. However, John interrupts Paul's display of his unusual achievement and asks him firmly, "What did I ask you to do? What did I ask you to figure out right now?" Paul reddens and replies, "To find out if you can split 100 cents in three people". John relaxes and says in his more normal, peaceful voice, "Ok, three people. That's what I want you to work on, OKAY? Not *flowers*".

John moves next to Jack, Paul's partner, who has just completed the problem. Jack has three piles of coins on his desk and a penny in his hand. When John asks Jack to articulate his process and his answer, Jack says, "Well, I divided the money into piles of 33 cents each, and I have this one left over, but I really want to give it to someone. I want to say the answer is 34, but I guess its 33 remainder 1". John does not comment on the difference in the two possible answers nor explore the tensions Jack feels in choosing between them.

Almost immediately after talking to Jack and Paul, John moves to a girl, Nancy. Nancy has shaped three flowers out of the 3 quarters and 25 pennies on her desk. Figure 1 is a reconstruction. She surrounded each quarter by seven pennies and placed an additional eighth penny on the top of each. All the quarters were seemingly positioned along an invisible circle, indicated by the broken line we have added, and with the remainder penny in the centre. Later, the teacher and the researchers referred to Nancy's coin arrangements as "flowers" in their follow-up discussions. As we watched the video recording, it was clear to us from Nancy's confident posture and smile when the teacher turned to her, that she felt she had successfully solved the problem.

Nancy has made three "coin flowers" each having the value of 33 cents with the remainder penny in the centre of the imaginary circle – almost like a whole pattern. John

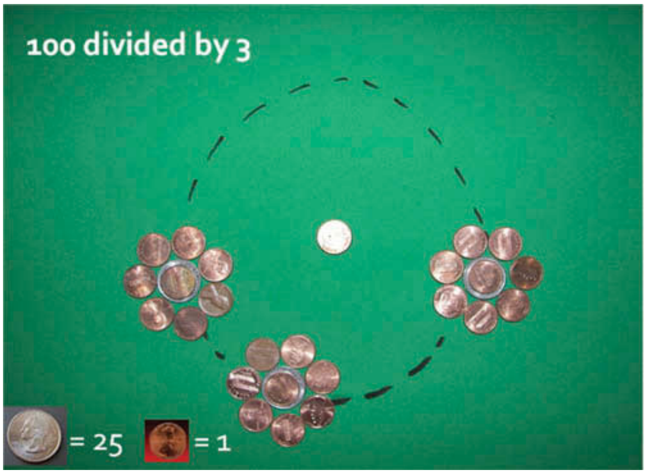

Figure 1. Nancy's aesthetically pleasing, topological solution.
Note: The photograph was taken from the video still frame; the broken circle line was added by us.

looks at Nancy's work, pauses and mildly startles. Nancy seems to have solved the problem correctly but in a way unexpected way to the teacher (and us).

John: I didn't hear from you, what did you come up with?
Nancy: [Startles] One extra . . . [Playing with the penny in the center with her fingers].
John: And how much did each person get?
Nancy: I didn't count that yet.
John: Can you do that right now? You just have to count one person's.

There was quite a prolonged pause (a minute and a half) and then Nancy counts on her fingers – 7 pennies around one of the quarters and one on the top – total 8 pennies. Touching the penny in the centre, she answers, "34", an incorrect answer.

Interestingly, just a few moments before, Jack, Paul's partner, had just stated it could be 34 or 33 r. 1 in the previous interchange with John. Jack also paused as he seemed to wrestle with who the legitimate owner of the remainder might be. But John accepted Jack's answer as correct. In contrast, he does not accept Nancy's answer. Instead, John tries to verbally guide Nancy to the right answer by using leading questions like, "What happens when you add 5 pennies to the quarter?" As he said later in an interview to us, he did so in an attempt at *scaffolding* her efforts to the correct answer of 33. Yet the teacher felt perplexed about why Nancy seemed to need guidance for "easy" mathematics for her from his knowledge of Nancy's mathematics skills. John next points at each group of one quarter with 8 pennies and counts, "Thirty-three, thirty-three, thirty-three, and this remainder, right?" "Right", replied Nancy. John, with this agreement, moves to another child, but as he said later, while watching the video-recorded episode, "I wasn't satisfied with the exchange".

Method

We call the school, CLC, a pseudonym used by Smith (2011) in his description of this same school. "CLC, the Creative Learning Center, a 90-student K-8 school located in a suburban town in the mid-Atlantic region of the United States" (p. 62). Visiting this school for research projects in the past, we have been impressed with the ontological nature of the staffs' teaching and concern for the meaningfulness of instruction for students. As a progressive and private school, staff are not required to gather standardized testing data on students, nor fulfil state-mandated curricular standards. The school staff is committed to a progressive and constructivist approach as part of its philosophy of educating the "Whole Child". In particular, we found John to be a consistently sensitive and caring teacher. In his nearly 30 years of teaching in this school, John had finely honed his classroom practices, curriculum and responsiveness to students.

In our initial research, John wore a hat with an attached video camera during the observed lesson. The hat camera allowed us to record and then to analyse the semi-private, personal interactions with students that often occur as teachers provide individual instruction. We also filmed the whole class from a second camera on the periphery, which provided us with a broader view and context of the interactions. We filmed again, using one camera, in our subsequent interviews with the teacher. In this review of the footage in our initial study, it was John who initiated the investigation into the present case as he stopped the scene with Nancy and noted his discomfort with the way the interchange had developed.

For this case, we held three interview sessions about the three interactions above. These also constituted data for our case study in a way similar to the Preschool in Three Cultures method (Tobin, Davidson, & Wu, 1989). In this series of interviews, we watched selected points in the footage that we had synthesized ahead of time for the teacher's comments. We recorded and transcribed what John said about all three interactions. We questioned John about how he analysed the sections separately and together, the influence of the student's actions on his decision-making, what broad pedagogical discourses he drew on in making his teaching decisions and if concerns related to school politics or parent responses affected the case. Ideally dialogic data analysis would include the student's reflections as well. Unfortunately, we did not immediately interview Nancy and did not show her the recorded interactions – we regretted this omission later.

Data analysis

Our unit of analysis involves the points in which student or teachers diverge on modelling of the word problem. We found three instances that together comprise our case. We included in our unit those broader pedagogical discourses that advise teachers about what to do with such divergences.

Our analysis proceeded in the following way: we transcribed each of three reference points of the teacher's interaction with Paul, Jack and Nancy as these allowed points of comparison in John's responses in the three sequences where the teacher promoted his interpretation yet was inconsistent in his acceptance of different aspects of the students' interpretations. We noted the ways in which the students' interpretations were diverted back to the teacher's (through reprimands, ignoring, suppression or verbal scaffolding). We also noted the teacher's supports for the students' interpretations; for example, the long wait times he afforded Nancy. What became interesting for us was how his constrains and supports of students related to broad pedagogical discourses. We wondered if these discourses in some way confuse or render the students' interpretive efforts to be invisible for John. In our interviews, he seemed to struggle to recognize the students' mathematical efforts.

We sought cultural validity for our study by placing the teacher's decision-making within the larger sociocultural context of pedagogical discourses likely to influence his practice. As Seaman (2008) suggests, "The researcher ought to seek data sources representative of events that occur 'out there.' Indeed, from an activity theoretical perspective, distal influences are very much 'in here' and are crucial to the understanding of an activity's internal structure and the positions actors take with respect to one another" (p. 9). We examined existing relevant literature in mathematics pedagogy that could have guided or influenced John's decision-making in this case. We suggest three major approaches that are relevant for understanding John's pedagogy: progressive constructivist, discursive scaffolding and situation modelling, and consider how a fourth – a dialogic approach – might have changed the outcome and addressed the issue.

The validity of our dialogic analysis of the cultural context of this case is defined by the coherence of our interviews within the sociohistorical backdrop of professional discourses that influence and stabilize broad pedagogical decision-making. Like Tobin and his colleagues in their study, we expect that our study does not represent the final word about the described phenomenon or even what defines the phenomenon, but we hope to engage readers in our dialogic research.

John's reflection

In our first interview with John, we asked him how he made sense of the mathematical problem. John said, "I could not make sense of how Nancy modeled the problem". As to his own sense of the math problem, he said he thought of it as a formula in his mind which looked like "100 ÷ 3 = 33 r. 1, as if on a kind of a mental whiteboard". John said, "I was thinking of the formula and how to make it understandable for my students". He suggested that when he developed the lesson narrative he meant to materialize the abstract problem. We noted his constructivist approach and choice of a "real world" setting.

We suggested to John the possibility that Nancy had made a topological model of the problem as an alternative to his own; one that used all the coins in a way meant to convey a whole design. John suggested this possibility made a lot of sense; Nancy was very artistic and often did things in a novel way. Her approach to the problem he had "never considered".

In our second interview, John admitted feeling frustrated with Nancy and his guidance of her, "It's almost like the [artistic] design she had made and the way I described the problem weren't connected at all. I felt, you know, that I wasn't using the right words or scheme or something to help her figure out what the answer is". Nancy seems, to us, to be in a different sense making "space"; evidenced in the way she startles[2] and looks puzzled by his question. Equally, John startles on coming to her pattern, indicating that he was in a different sense-making space.

John continued, "I left dissatisfied, because, you know, Nancy often comes up with her own unique ways of doing things". He suggested that he even sensed that she didn't need guidance because she had made a mathematics error. He became irritated with himself. He suggested that he felt he had failed to be able to articulate for himself exactly what she had accomplished and this lead to him feeling that his guidance somehow missed her learning needs.

John told us that in the past Nancy had a way of thinking differently, and this had often proven fruitful for his guidance of her learning. Yet, in reviewing the video clip, we noted that he did not ask Nancy about her thinking, though John had the luxury and even encouragement from other teachers, parents and students to follow his student's thinking in this school context (unlike the curricular constrains often placed on public school teachers in the US). John also did not seem interested in reflecting on Nancy's thinking with us. Thus, we suspect that something other than institutional and classroom constraints affected his orientation to Nancy and promoted his apparent lack of interest in her unconventional way of solving the problem though he felt tension and ambivalence about his teaching of her.[3]

Our interpretation

Nancy's startles and long pauses seem to us to express important inner struggles for meaning as her embodiment of the problem is re-interrupted by John. We use "embodiment" rather than "model" at this point to indicate that her model becomes part of a lived reality for its author – a way of her being in the world and with the mathematics at the moment. In the 1:30 minute long segment, Nancy seems suspended in time – perhaps caught in the semantic struggle space between her interpretation and the teacher's representation – and then, as their interaction shifts onto the plane of John's interpretation, she seems suspended again, perhaps developing new questions and interpretive problems similar to those of Jack.

John's fairness problem setting is arguably polysemic – but rather than exploring its polysemic features with students, he uses his narrative as a launching pad for his monosemic symbolic representation of the problem, without much thought about alternative interpretative frames of the math problem and the social situation of fairness (see Table 1). Conversely, the students appear to be concerned with how the mathematics story would unfold in a real setting, raising questions of social fairness especially around the what to do with the remainder. When we pointed out Jack's concern to John, he replied, "Oh, I've heard students say that for 20 years and I knew the remainder was confusing for them, but I never understood how". It was interesting to us that he had never explored the students' confusion about the reminder and the issue of fairness.

We wondered what was it about John's mathematics pedagogy that prevented him from inquiring about Nancy's model, or the recurring question of fairness that students had raised, even though these issues obviously had caused some tension and frustration for him and his teaching for some time.

We wanted to understand the sociocultural context that influences John's pedagogical decision-making and potentially even guides his attention, his curiosity and his articulation of student problem interpretations.

Interpreting the case within prevailing pedagogical discourses

We next shift from our case for a moment to discuss four prevailing broad pedagogical discourses in terms of what kind of guidance each provides for a teacher in making sense of students' unconventional interpretations, and what room each provides for the interpretive capabilities of students. These are *the progressive constructivist approach, the scaffolding discursive approach, the situation modelling approach* and *the dialogic approach*. We then discuss the way in which John appropriates some combination of the first three approaches, and then define how these approaches fail to help him make sense of Nancy's model. We then propose how a fourth – dialogic pedagogy – approach might have successfully attuned John's perception and decision-making to make valuable use of Nancy and other students' mathematical interpretations.

Progressive constructivist approach

The progressive constructivist approach, as we label it, is rooted mostly in the works of two important theorists: Dewey (1902), for his influence on progressivism, and Piaget (1950), for his influence on constructivism. Dewey (1902) suggests we ought to "double psychologize" the curriculum preset by the society. That is, make society's preset curriculum accessible to students through reconstituting the culturally given, socially important ideas as much as possible: (1) through experiences, from which the ideas were originally historically derived, and (2) by connecting the students' personal experiences and interests to the broader curricular ideas of society. According to the constructivist thought, which has its origins in Piaget (1950), knowledge cannot be simply verbally conveyed to the child by adults but must be enacted by the child through sense-making activity in order that the child reconstructs knowledge along an internal logical schema. In contrast to Dewey, Piaget did not insist on preset curricular end points, but rather, expected that the students' constructive learning activity would be unavoidably guided by the logic of the process and the natural laws of the world.

These natural laws of the world would be reflected, according to Dewey, in society's curriculum. He saw the problem with students' meaning-making due to this curriculum enacted in a way that strips out the layers of discovery and meaning-making that initially occurred for scientists in making these discoveries. Teachers, in his view, must endeavour to make learning *progressively constructive* by grounding their lessons with preset curricular end points in "real world" word problems in a way that is likely to connect the everyday experience and prior knowledge of the children with a problem context that relates to its authentic use.

However, in criticism of how this pedagogical approach is often realized in educational practice, Lesh, Doerr, Carmona, and Hjalmarson (2003) suggest that what the teacher often expects of students does not represent the student's own construction of the problem to unique ends. The presented problem is not expected to lead to creative playing around with materials, develop socially problematic situations or create alternative possibilities for defining the problem and new corresponding mathematics models.

We hear the progressive constructivist discourse in John's justification of his mathematics lesson and guidance of the students in the case. As he suggested in an interview, "I wanted to make the problem real for them". Thus, he did not want to teach his students about division with a remainder through direct instruction (Hunter, 1971) and rote responses, rather, he wanted the students to reconstruct the culturally given mathematics practice through their idiosyncratic hands-on activity. He expected that this would, logically in this discourse, lead students to discover the same formula he had had in mind in advance.

Discursive scaffolding approach

With the sociohistorical framework of Lev Vygotsky, proponents of a discursive scaffolding approach, in general, understand learning as (1) participation in "shared ways of using language, ways of thinking, social practices and tools for getting things done" (Mercer & Littleton, 2007, p. 3), and (2) the growing independence of the learner under the guidance of a more knowledgeable peer or adult (cf. "the zone of proximal development (ZPD)", Vygotsky, 1978). Though, as Chaiklin (2003) has suggested, Vygotsky himself might not necessarily have recognized his concept of ZPD within this approach. Bruner and his colleagues (Wood, Bruner, & Ross, 1976) interpreted Vygotsky's notion of the zone of proximal development as a special discourse between a child and an adult, called "scaffolding", that bridges the child's incomplete, confusing and even misleading understanding of the task (and the world) towards adult mastery and understanding (i.e. culturally share interpretations). A number of mathematics education researchers, working within this framework, argue for a discursive approach to mathematics instruction (Ball, 1991; Horn, 1999; Lampert, Rittenhouse, & Crumbaugh, 1996) because (1) it permits students to discuss apparently faulty mathematical reasoning without the teacher jumping to evaluations of "right" or "wrong" and (2) it recognizes that how people think about mathematics is layered into the way they derive solutions. Lerman (2000) argues that: "Children become mathematical by getting used to what counts as being mathematical, which is constituted in the social practices of the classroom. This may be a more fruitful way of speaking about learning, in which learning is about speaking, about how to speak in the legitimated codes of school mathematics" (p. 50). Finally, (3) the discourses, values and, especially, identities formed within discourse communities are crucial to becoming knowers because these are important in the service of students arriving at cultural ways of speaking and solving mathematics problems. In a discursive scaffolding approach, students are expected to arrive at the culturally given practice through peer argumentation and teacher's skilful scaffolding (and orchestration) leading to a conventional solution.

However, Sfard (2008) critically argues that the peer approach is meant to discover sociocultural conventions not the inner logic of the discourse: "The learner would be prepared to follow a rule enacted by another interlocutor as a prelude to, rather than a result of, her attempts to figure out the inner logic of this other discourse" (p. 250). As Sfard argues, where conceptual narratives fail to resolve, the learner should adhere to the teacher's model, "More specifically, the discussants need to be in consensus with regard to whose discourse should be the model to follow, they have to act according to their respective roles of learners and teachers, and they must have a unified vision of the goals of learning and of the course this learning is going to take" (p. 292). In other words, the student must participate initially as an outsider to herself, imitating the teachers' way of putting words to "deeds", as Sfard calls students' construction of word problems. These words-to-deeds (what Sfard calls "rituals") later become the basis for the student recognizing the salient features of these rituals and being able to successfully translate these to other mathematical problems. The student is considered to first perform the deed on the social plane and then internalize the process and the focus is on the adult meaning and purposes.

There are other sociocultural researchers who have suggested students' meanings should, if not must, prevail in guiding instruction. For example, Aukerman suggests, "A notable departure from this trend is Rogoff's (1990) term *guided participation*, which she explicitly links not only to joint activity, but to joint purposes. She maintains that an adult cannot guide participation unless the child to be guided shares a common purpose. Here, the assumption is that the child and adult have a collaborative goal. That is, the teacher's desire is to support the students' development of their own meanings and not simply convey the culturally given meaning, but the instructional means of doing so have been problematic. *Guided participation* has not taken hold in the educational vernacular to nearly the same extent as *scaffolding*" (2007, p. 63).

The discursive scaffolding approach aids students' interpretive powers in that it promotes their participation in culturally valued practices, guiding students to internalize conventional models in a discursive manner. Yet, as the goal of instruction is towards a socially constructed interpretation, it is ultimately a standardized orientation. Scaffolding student understanding normally privileges the teacher's representation and interpretations

(Aukerman, 2007). For example, we saw that Jack has become adept in his ability to skip his own interpretation questions and align the problem solution with the teacher's intended abstract equation, re-conceptualizing the problem in terms of the teacher's algorithmic discourse.

The discursive scaffolding approach to interpreting problem situations with its focus on cultural reproduction through quick arrival at consensus is criticized:

> Because children do not constitute a mathematical community, the teacher is seen as the representative of the mathematical community.. . . The purpose of the discussion is to smooth out the individuals differences in understanding the situation and bring them more quickly into line with conventional ideas, methods, and symbols and language. (Lamon, Parker, & Houston, 2003, p. 446)

In John's teaching practice, we see aspects of a discursive scaffolding approach in his encouragement of his students to work together and discuss the problem and its possible solutions. Nevertheless, he considers there is unified agreement of the problem that is being defined and solved.

Situation modelling approach

The situation modelling approach seeks to encourage teachers to deeply develop and build upon students' interpretations of the problem situations to address difficulties that many students face as a result of overly decontextualized mathematics instruction. Rosales, Vicente, Chamoso, Muñez, and Orrantia (2012) suggest that there is not enough focus on the situational aspects of word problems in mathematics instruction, "few studies have analyzed how situational and mathematical models construction is promoted during word problem solving in day-by-day classroom practices" (p. 1186). Rosales et al. (2012) investigated what kind of instructions teachers use when given non-standard word problems because these inherently create a greater need to focus on modelling the situation. They asked, "When mainstream teachers solve non-standard, situationally and mathematically reworded problems jointly with their pupils in their mainstream classrooms, will they: Use a paradigmatic or narrative approach? Promote a deep mathematical processing of the problem? Promote a deep qualitative comprehension of the situation described by the problem?" (p. 1187). They found that the teachers' "approach to problem solving was mainly paradigmatic, even though our experimental problems were non-standard" (p. 1192). That is, the teachers' training and use of available pedagogical discourses seemed to promote their focus on standard, decontextualized, problem solving and not the situation interpretation. In the Rosales et al. (2012) study, a few teachers did focus on the interpretation of situation, but "the teachers could not resist the temptation to process all the information contained in the problem and they also made public the irrelevant situational knowledge" (p. 1193). In other words, the teachers focused on their own interpretations of the problem situation, cancelling features non-salient to the expected student interpretations they hoped to elicit.

Another set of researchers focus on students' modelling of the problem situation within "a community of learners" (Brown & Campione, 1998) and draw on both of the previous approaches:

> Drawing from the Vygotskian tradition of thought as mediated activity (Wertsch, 1991), the models and modeling perspectives also emphasize the roles of conceptual tools, such as those that are supported by language or notational systems, that influence the power of peoples'

> thinking. Because of the power of such conceptual tools, they generally have strong influences on students' mathematical thinking from both an individual endeavor and a collective enterprise. (Lesh & Lehrer, 2003, p. 121)

The emphasis on situation modelling draws on the progressive constructivist approach as it recognizes that concepts include many layers of tacit and explicit information embedded in problems, as they are, themselves, sociocultural constructs. It draws on discursive strengths as it realizes that mathematics problems represent a speech genre that students must be able to make sense of and use to communicate their own ideas to others. This approach invites students to develop and try multiple models, whether idiosyncratic or not, though eventually the teacher will lead them to understand the conceptual power of "big ideas" in communicating with social others.

While students' interpretations of the problem situation context are welcomed and promoted by both situational context and math-modelling researchers, teachers in the situational context try to guide students to conventional interpretations, and researchers in math-modelling express discomfort when students produce unconventional and idiosyncratic interpretations. They guide teachers to steer their students towards generally useful interpretive ends, i.e. "Big Ideas". Though both are much more focused on students' interpretations, and the allowance and even promotion of idiosyncratic ones, the ideal conceptual interpretation in this approach is considered to be the one with the most conceptual and generalizable power. This represents the end point of instruction. This value narrows the possibility for students to express and experience the value of their own interpretive power alongside more general ones. This can, in our view, tend to suppress the meaning-making students may make of a problem that is strongly useful for themselves (i.e. high conceptual power for a few persons rather than generalizable to all situations), risks filtering out particular cultural perspectives and dissuades the creative capacity for innovation about mathematics that students might attempt to express.

John's use of a "real world" word problem seems to be guided in some ways that are similar to the situation modelling approach, as he dialogues with students about the interpretive contexts of the problem above and beyond his use of constructivist materials, and much like the few exemplary teachers in the Rosales et al. (2012) study. Along with proponents of the situation modelling approach, he seemed to believe that the social situation of three friends dividing a dollar would naturally promote in the students emergence of the mathematical formula he wanted to teach. He seemed to believe in "the mathematical isomorphism" – the presented social situation and his mathematics formula naturally have the same underlining mathematical relations. Despite different material, the word problem and the formula are "shaped" the same. In other words, the teacher seemed to believe that a well-articulated social situation legitimately promotes only one mathematics model (or even a limited number of the legitimate mathematics models). As we see in our case, this assumption was not true. Nancy managed to develop an alternative mathematics model out of the apparently same social situation. And, as we will show below, there can be infinite number of such legitimate alternative models.

Summary analysis of first three approaches

The three approaches discussed above all assume that the students should arrive at expected curricular end points, whether through discursive scaffolding or through students' own idiosyncratic, experiential and situational interpretations. The role of teachers is to act as representatives of the curriculum and guide students to these ends (i.e.

curricular standards, see the Common Core State Standards in the US, National Governors Association, 2010). Students are positioned by these pedagogical approaches as cognitive apprentices to conventional mathematics practices (Rogoff, 1990), while education is defined as cultural reproduction (Bourdieu & Passeron, 1990). Teachers are expected to guide students' learning participation by drawing on differing instructional methodologies to help students reproduce conventional meanings in culturally valued practices. Often, teachers see their role as helping students bridge the deficit gap between the conventional knowledge of the culture and students' incomplete knowledge and misconceptions: "A deficit gap is assumed to exist between scientific thinking and the child's actual thinking and this sets a developmental goal for instruction ('the zone of proximal development' [ZPD]) and the curriculum. The ZPD represents a lack of mutual understanding that is viewed by Vygotsky as negative (i.e. undesirable) and temporary" (Matusov, 2011a, p. 102) at least according to the understanding of Vygotsky that is promoted in the US. Along with, or perhaps because of, this focus, students' questions about side issues, "off-topic" remarks, alternative interpretations of the problem and playful exploration become problematic for instruction "off-script" (Kennedy, 2005).

We suggest that students develop alternative interpretations, even when discouraged from doing so. We think this is due to several reasons. First, students have their personal webs of meaning to draw upon, so that people's lives develop different understandings about what is salient in a given problem context, as Bakhtin (1981) suggests, "the meaning of whatever is observed is shaped by the place from which it is perceived" (p. 21). Thus, Nancy sought to bring her aesthetic topological orientation to the problem – a design skill with increasing economic importance (Pink, 2006). Second, students' involvement and engagement in a math dilemma opens the opportunity for their unique personal improvisation and creativity. Thus, for Paul, the materials present interesting possibilities to play and experiment with, which may (or may not) be relevant to the problem at hand, but from Lave's (1992) perspective may develop the dilemma that needs a solution. Third, it is simply exciting to try to develop ideas that differ from those of others. The life of people is not so much in the reproduction of, but rather, in the production of culture through authorial transcending of the culturally given, i.e. "authorial agency" (Matusov, 2011b). We suggest students will always develop unconventional interpretations no matter how carefully teachers try to reduce this possibility and the attending diversions from expected outcomes this creates. We suggest that such diversity is actually desirable. We even suggest promoting students' active reconstruction of the given cultural practice without regard for the student's own meaning-making, though active, is actually anti-cultural and even anti-educational.

We turn to discussion of the dialogic pedagogical approach that, in our view, might have addressed the pedagogical breakdowns that John faced in his class as it promotes students meaning-making as well as conventional knowledge.

Dialogic pedagogy approach

As we have suggested, it is difficult for teachers to control the themes, meanings, etc., that emerge for students. In fact, we think it is not even desirable to do so because in our view this would suppress students' curiosity, authorial agency and engagement in problem solving, and cuts short the opportunity to explore differences among student interpretations. We suggest, along with other researchers in the dialogic approach, that it is more beneficial for the teacher to expect and support this gap: "She [i.e., the teacher] is open to her students' ideas and is willing to pursue unexpected approaches to generate new mathematical understanding – the core of dialogic discourse" (Knuth & Peressini, 2001,

p. 325). Students and the teacher naturally *complexify* the problem by their own interpretations and these diverse meanings develop into provocations for each other, which aligns with natural human activity, "A gap in the mutual understanding between people is a necessary condition for dialogic, human communication, and for the entire human relationship" (Matusov, 2011a, p. 103). These gaps often emerge as *heterodiscoursia* – multiple discourses legitimately occurring at the same time as an effort to develop rich meaning. Heterodisoursia naturally involves *interaddressivity* – mutual interest in each other's diverse thinking, understanding and world view (Matusov, 2011b) not only between peers but also between students and teachers. We want to develop these two ideas further as we find they are relevant to an understanding of the dialogic approach.

Heterodiscoursia and interaddressivity

Heterodiscoursia (paznorechie, "разноречие" in Russian) refers to legitimate simultaneous diverse discourses (Bakhtin, 1986; Matusov, 2011a). Multiple discourses in our case emerge in discussions about the problem situation, but are often viewed as either "off-task" (Paul's fooling around with "flowers" in the teacher's eyes) or "on-task but off-script"[4] (Nancy's aesthetic discourse in the mathematics class). We suggest that Nancy's excursion into aesthetic form was valuable math-making, relevant to the classroom mathematics problem. Aesthetics discourse appears to be generative of mathematics here, and is the reason we take Paul's "goofing around" seriously. Allowing for heterodiscoursia in the math class dialogue supports aesthetics, social relations, ethics, personal experiences, etc., into the space of the mathematics lesson and promotes the participants' mutual interests in each other's emerging frames (i.e. interaddressivity). Research in museum studies (Matusov, 2011a; Palmquist & Crowley, 2007) indicates that when learning takes place within heterodiscoursia participants are able to bring their deep and diverse questions, purposes and meanings to the discussion through their emerging interaddressivity. This mix of discourses, connections, personal meanings and authorial interpretive insights yields conceptual dividends, comprehension depth and enhanced engagement, although it often makes conventional teachers nervous – afraid to lose thematic and communicational control over the classroom (Kennedy, 2005). Heterodiscursive conversations, then, are fruitful sources for students to build diverse understanding of the problem through playful exploration, mathematical modelling, the testing of diverse understandings and ideas, etc.

Not only are diverse discourses useful to explore the meaning in a problem, but multiple actions are also important to discovering diverse meanings. Unless authoring new definitions of the problem in multiple frames recursively, people may often not fully and deeply understand how to solve a problem. New problem definitions and models emerge on the periphery of people's actions while they try to define the problem through "playing" with objects and ideas (cf. John's characterization of Nancy's and Paul's "goofing around"). As Hegel argued, an individual "cannot define the goal of his action until he has acted …" (cited in Leontiev, 1981, p. 62). In other words, the problem definitions are usually not fully known until the problem is defined, means are found and the situation is formed. In fact, often a person does not even know what his or her frame is until he or she starts defining a problem and looking at the solutions! For example, while Paul was experimenting with the coins to develop an aesthetically pleasing pattern, which John condemned (in his case) as a "flower", Paul, arguably, had not yet developed Nancy's more complete frame of an aesthetically pleasing model within the required stipulation of the problem (using only $1's worth of coins). Paul might have been on his way when John interrupted the emergence of Paul's free-play/frame-experimenting boundary. John's isomorphic orientation (i.e. the formula in the teacher's mind and the problem narrative are isomorphic to each other) prevented his, the teacher's, realization of the value of alternative problem definitions and frames that students might develop in contrast to his own. We would like to highlight the importance of playing with available materials as Paul initially attempted to do. "This is how mathematics problems arise – just from sincere and serendipitous explorations. And isn't that how every great thing in life works?" (Lockhart, 2009, p. 138). Saliency emerges from playful exploration and, with it, related calculations.

We want to suggest that students' idiosyncratic understandings are always important in students' local classroom communities of mathematics practice. To highlight these diverse understandings, in Table 1, we conceive of ways students might approach the case problem of dividing $1 by three people. We do not presume to present an exhaustive list as we suspect there could be any number of interpretations. We think, though, that by developing a short list of possible diverse interpretations and their frames:

(1) We raise awareness that student and teacher can be in differing semantic spaces.
(2) Teachers will be better able to assist students in articulating their ideas if teachers have a sense of the diversity of possibilities students might raise.
(3) Teachers can develop greater awareness of his/her own modelling propensities and be better able to question his own tacit assumptions.
(4) How students frame the problem as they discuss their models, social questions and conceptualizations determine what becomes salient for the solution. What becomes salient then guides the teacher's decisions for providing procedural knowledge.

These diverse interpretative frames, defined (or even embodied) within diverse social and discursive practice contexts, are not always or fully mathematical as students' tacit responses to social situations might more heavily guide their thinking initially. Some of them do not need mathematics at all (see, e.g. the VIP frame in Table 1). This non-exhaustive list of possible representations further reveals the complex, situated and diverse orientations that word problems can have with their representational models:

Table 1. Diverse heterodiscursive problem defining and problem solving of 100 ÷ 3.

Heterodiscursive frames	Explicating narratives of social relations and situations behind the frames
Accuracy algorithm	The solution of a mathematics problem follows a sequence of steps well defined by the teacher and professional mathematicians, and trusted by the students. The correctness of the algorithm is checked by its accuracy – following the algorithm exactly (as it is done in computers). The remainder is not problematic.
Mathematics isomorphism	Real mathematics is not in the example, but in the abstracted formula, salient features are already captured by the mathematics model with which one must work. As soon as the mathematics model is abstracted from "a real life case", the corporeal nature of the problem must be abandoned. The abstracted mathematics model affords the solution through previously solved mathematics model problems (axioms and theorems). The correctness of a solution is checked by mathematics proof discourse independent of any "real life case". The remainder is not problematic.

(*Continued*)

Table 1. (*Continued*).

Heterodiscursive frames	Explicating narratives of social relations and situations behind the frames
Fairness	"Divide $1.00 among you and two friends" (John's frame). The correctness of the solution is checked by the egalitarian friends' sense of peace based on perception of the coin quantity sameness. The remainder is problematic.
Sharing	"Three friends share a dollar and buy a ball to play with together". The correctness of the solution is checked by the friends' successful joint activity mediated by $1 tool (i.e. having enough money to buy a ball). No problem with the remainder even if there can be leftover sum from buying a ball (if the ball costs less than a dollar).
Sufficiency	"I'm going on vacation for a month. For my medical condition, I have to take 3 pills a day, I have 100 pills, do I need to order a refill, or is it enough?" The relevant focus is on how many days I have pills for. The correctness of the solution is checked by the person having enough pills for the vacation. No problem with the remainder.
Neediness	"Find out who is the neediest and give this person the available amount of money". The neediest person gets $1.00, or a big chunk of it. The correctness of the solution is checked by the defining the neediest person, and whether the available amount is enough to address the person's needs (if the amount is more than enough, the rest may go to the next neediest person). There is no problem with the remainder unless all needs of the involved people are covered and there is still leftover.
VIP	Find the most important person in the group (e.g. the youngest, the oldest), the only girl ("ladies are first") and give $1.00, or a big chunk of it, to this VIP. The correctness of the solution is checked by the defining of the VIP, getting a consensus on the definition and who is VIP, and how much to give to VIP. No problem with the remainder (as it always goes to VIP). No need to count.
Aesthetic pattern	Develop and aesthetic pattern that is symmetrical and pleasing to the eye among three centres and constrained by $1.00 (Nancy's frame). The correctness of the solution is checked by the visual symmetry and beauty. There is no "remainder" in the solution. No need to count.
Pleasing the teacher	Follow the teacher's correct procedures without trying to make sense of them. Try to grasp the pattern in the teacher's actions with whatever embodied means you have (e.g. mnemonics). The correctness of the solution is checked by whether the teacher is pleased. No problem with the reminder.
More…	We suggest by more that we cannot and probably should not attempt to delineate all possible mathematical frames as dialogism is open to perspectives that may not yet be imagined.

Each of the different embodied frames generates different possible interpretations and calculative challenges (if at all) and usually requires different mathematical models to mediate them (if at all). We think it is important and useful to understand the multiple interpretations students might have. However, we suggest that – while recognition, publishing and documenting possible models within a professional teacher community is very useful for the teacher's awareness of this diversity and improvisational guidance – the number of diverse models is as unlimited as the number of practices, as student and teacher interactions, personal experiences, materials, situations, creativity, sociocultural relations and practices, and future changes are likely to be unlimited. The teacher needs to develop an interest in the infinite diversity of possible embodied interpretative frames and the thinking of his/her students (i.e. interaddressivity; Matusov, 2011b).

We imagine a dialogue with these ideas in mind and have placed it in the Appendix, recognizing that such a dialogue lacks the life of real children and teachers. We use it as a means to illustrate how including heterodiscoursia and interaddressivity into a mathematics lesson enriches it. As students ideas begin to unfold, their conversation will naturally fall out of the bounds of "math". Yet, doing so is necessary for encouraging emerging ideas and discovering what is salient for whom and why. Students will address each other's ideas, and the teacher can gather these to a local board, which, in itself, will create the need for students to develop their ideas further as they attempt to share them in dialogue. Questions can be asked such as, "How will we model this situation? Are these models legitimate: for what and for whom? How do these models relate to each other? Do we need mathematics to solve it, or can mathematics provide a shortcut way to solve it? What other situations can this solve? What new questions do we have? What other ways do people use to solve these kinds of problems?" These ideas can simmer on a display so that students can congregate and discuss their ideas further. In this approach, all voices bring something of value and the power of interpretation is shared among members of the classroom community as well as outside of it.

Issues of power

When the teacher's interpretation is privileged, students become adept at sleuthing out what the teacher, text or standardized test "wants" in order to generate the problem structure and solution for which they will receive credit (c.f., Heyd-Metzuyanim, 2013 for a painful example of this process). This often necessitates that students disconnect from their ideas, interpretations and (potentially) their excitement at developing their own meanings. Instead, they must cypher out the hidden rote task embedded in the problem situation. Institutionally sophisticated and savvy students learn to replace the task to mathematize one's world with the school task of how to think like the teacher, often "by taking the data of the problem and selecting the algorithm to be employed using some meaningless strategy based on some salient element in the problem, like the key word strategy" (Rosales et al., 2012, p. 1186). Students become adept at negating the interpretation process because it is not really wanted.

Conversely, our case shows that students in a low authority context freely choose to interpret word problems rather than resort to "schoolish" strategies and will deviate from the teacher's original problem to explore the potentials and meaning presented by the problem. Often in education the teacher's interpretation of the word problem has preeminence, and his power then defines what is considered to be a successful interpretation of the problem by the students. In the dialogic approach – the individual meaning-making behind the mathematizing is preserved. People use math in different ways: to give voice to different concerns, to define differing sets of relations, to relate to their world through different lenses and to make mathematical meaning of our world. The dialogic approach allows students to experience their capacity, and those of others, to use math in this way.

Conclusion

We began by describing the case in which a student developed an alternative interpretation of the situated problem context for a word problem within a third- and fourth-grade mathematics lesson. The case is interesting on many levels (for instance, it reveals some of the ways students think about word problems), but we focused on how broad pedagogical discourses promote or constrain students' thinking. We think this issue is important in order to better research students' mathematical meaning-making. Because of the relative freedoms for interpretation afforded students by the progressive schooling context, we were able to study the teacher's decision-making primarily in terms of prevailing pedagogical discourses influencing his ideas and note how these guide his values, his relationship with students and his interpretation of students' ideas, and even, seemingly, the perceptions he had of the students' actions. We suggested four categories of pedagogical approaches that either illuminate students' meaning-making for teachers or seem to filter it out, even when the teacher's sense-making picks up on students' legitimate alternative efforts at modelling. We found each approach tended to mediate the teacher's focus on some aspects of the problem context, but rendered other aspects invisible to the teacher's perception (or at least his articulation of these perceptions). We think further research is needed to explore this possibility.

The orientation of the first three approaches assumes the teacher represents or "transfers" important cultural practices to students. This means (1) that students who desire to connect the learning to their unique perspectives and experience are often prevented from doing so, and (2) original and idiosyncratic solutions and methods are, if valued, instrumental for reaching forgone conclusions.

The heterodiscursive dialogic approach promotes students success in cultural practices in a way that is subjective and authorial – a participant authors the problem dilemma based on his/her web of personal meaning and goals in the particular contexts. Students have legitimate subjective and authorial roles in their learning and in the mathematizing of their world (developing their own representations for their own purposes).

The authorial orientation of the dialogic approach supports and promotes students' development of culture making (i.e. culture production, transcendence of the given) rather than (only) socialize the student into pre-existing cultural practices (i.e. reproduction of culture, arriving at the preset curricular end points). It recognizes students as creative and active participants in culture-making, and also that culture is continuously emergent – a view consistent with an emergent view of epistemology – meaning student interpretations can have legitimate epistemological value. Thus, the dialogic approach supports students' legitimate participation in the culture as well as their legitimate role in helping culture to transcend itself (Matusov, 2011b).

In dialogic pedagogy, teachers remain epistemological learners (Matusov, 2009) of the academic subject matter at hand as they consider student's novel experiments and embodied frames. This allows institutional roles and hierarchies to be temporarily suspended (without being abandoned), as the relationship between teacher and student becomes collegial (about the math) within a pedagogical regime of "internally persuasive discourse" (Bakhtin, 1981; Matusov & von Duyke, 2010) in which "truth becomes dialogically tested and forever testable" (Morson, 2004, p. 319). We suggest the greater goal of education is for students' critical interpretation of the world and their communication of their ideas: a set of skills useful to the life of the child and into their adulthood in all content areas (and even outside of content areas).

Disclosure statement

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Notes

1. The names of the teacher, school and the students are pseudonyms.
2. We want to thank James Cresswell for pointing this out to us (J. Cresswell, personal communication, 10 October 2010).
3. We provided a copy of this article to John for his review, and he agreed with our description and analysis of the case. Even more, he found this process so fruitful for his own teaching, he later asked the school staff to consider video as a way to reflect on their practice together as a whole.
4. See Kennedy (2005) for her discussion of the "on-task but off-script" teaching phenomenon.

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Appendix. Imaginary alternative guidance in a heterodiscursive dialogic approach

We tried to re-conceptualize the lesson presentation below by imaging it as a classroom dialogic pedagogic discourse that includes heterodiscoursia, interaddressivity and the teacher's support of students' emerging interpretation of the problem in order to illustrate how these features work together and function at the classroom level, i.e. to reveal diverse understandings, bring them in dialogic contact and test them against each other. In John's initial conversation with Nancy, in his circulating to assist students, he first explores her different frame, unfamiliar to him, simply to appreciate her thinking:

(1) John: I didn't hear from you, "What's happening here?"
(2) Nancy (smiling with satisfaction): I made this…
(3) John: What an interesting pattern! It is very different from what I came up with to solve the problem. I'm not sure I fully understand it. Can you explain it to me please? What did you do?
(4) Nancy: I got $1 in coins, and I divided them. It looks even.
(5) John: So, did you solve the problem?
(6) Nancy: Yes, I divided $1 by 3. I made it looking nice.
(7) John: And what is the answer?
(8) Nancy: All these circles, this pattern. The coins fit together.
(9) John: These flowers?
(10) Nancy: Yes, these three flowers.
(11) John: You just expressed something really interesting! (By now, Jack, and Paul, and other students have begun listening.) But I was expecting you to have a mathematical solution.
(12) Nancy: well, I used the money you gave us, and I divided it…
(13) John: Hmmm.… It seems to me that in your solution, the pattern itself solves the problem of division in a different way from mine.
(14) Nancy: Yes, because I used the same amount of money that you gave everyone else, I even divided it three ways. Did I do it wrong?
(15) John: No! I think you're right! How interesting! In your way, we don't need to know how much is in one pile or if there is a remainder, do we?
(16) Nancy: No, it's a design that uses all the parts.
(17) John: The visual pattern is the solution. Remember, we talked about patterns in our class? It even looks like the whole thing is a design that could be a flower itself.
(18) Nancy: Oh yea, I see that now, I wanted to add more coins! At first, I just tried to make it beautiful.
(19) John: What makes it beautiful to you?
(20) Nancy (smiled): I don't know. The flowers are beautiful. It's like drawing a beautiful picture.

(21) John: What do you think makes flowers so beautiful?
(22) Nancy: Maybe, because they are even from all sides (placing her hand on a "flower" of coins as if setting different directions).
(23) John: We call that "symmetry," remember? We talked about that.
(24) Nancy: Oh, yes, symmetry. Flowers are symmetrical and my big circle is symmetrical and this one (pointing at the "remainder" penny) is the center of it all.
(25) John: Nancy, this is a great solution! How did you decide to take 3 quarters and not dimes, for example?
(26) Nancy: I don't know, I was playing with coins.… I just took $1 worth of coins and you suggested they should go to 3 people, so I thought of 3 piles.
(27) John: Ok, what if you have to divide $2 for 3 people?
(28) Nancy (thinking for a while): Oh, this is rather simple. You need just to double the coins. The circle and the flowers will remain the same but they will be thicker. Or maybe I'll do more flowers…
(29) John: Can you solve the problem without quarters but only with pennies, nickels, and dimes?
(30) Nancy: I don't know. I need to play with the coins. John, how did you solve the problem?
(31) John: Well, I can think of many ways, but in simple way, I would select 100 pennies and then start giving one by one to each person until only one will be left.
(32) Nancy: And what would you do with this one?
(33) John: Nothing. It's the remainder.
(34) Nancy: Would you throw it away?
(35) John: No.
(36) Nancy: Who will get it?
(37) John: I don't know, I have been wondering about that. What would you do with the left over in your solution?
(38) Nancy: I don't have any leftover. The penny in the center is a part of the beauty.
(39) John: Yes, but who will get it?
(40) Nancy: Nobody. It's a part of my pattern.
(41) John: Wow, this is an interesting discussion. I think we need to discuss this in our class. It's interesting to find out if other folks would agree with our solutions. I wonder if other students solve the problem different from you and Jack, and me and how our approaches are connected.
(42) Jack (to Nancy): I don't think it counts as mathematics if it doesn't have numbers.
(43) Nancy: Jack, math isn't only about numbers. It can be also about beauty, about patterns, and about symmetry.
(44) John: Let's discuss that with other kids as well. I wonder in which situations people might need to solve problems like a pattern and when they would need to figure out what to do with the remainder, and when my solution is the most useful, or even what could change in the world that would make your solution more generally needed? Also, we might think about what is math? Does it always deal with numbers? Very interesting!

Please notice, especially, line 41 in which John invites the class to use Nancy's interpretation as a tool for thinking about their own ways they interpret and represent the problem situation. Nancy's very different modelling of the problem would highlight the difference between modelling the situation and its mathematical representation. In our view, her alternative vision develops a very interesting lesson for the class important for students' sense of mathematizing their world.

We envision that this dialogue might take a few class periods (if not further in the participants' future) to develop deeply, first by bringing the frames before whole class (some of whom would have overheard and been watching their discussion) and then, have Nancy and perhaps a few others, summarize their thinking and think about how mathematics models might be communicated to others. Students could compare and contrast the presented models, whether finding a remainder is important in each model, if knowing the amount in one pile is needed or not, and relate each to the mathematics notation used to express it. Importantly, in a dialogic approach, Nancy's interpretation of the word problem would not be rendered invisible for John's decision-making for how to guide Nancy. It would be welcomed and explored, rather than redirected. It would become a rich resource for class discussion and meaning development.